Logical Consecutions in Discrete Linear Temporal Logic

V.V. Rybakov

Department of Computing and Mathematics
Manchester Metropolitan University
John Dalton Building, Chester Street, Manchester M1 5GD, U.K.
V.Rybakov@mmu.ac.uk

Keywords: temporal logic, linear temporal logic, logical consequence, inference, inference rules, consecutions, admissible rules

Abstract

We investigate logical consequence in temporal logics in terms of logical consecutions, i.e. inference rules. First, we discuss the question: what does it mean for a logical consecution to be ‘correct’ in a propositional logic. We consider both valid and admissible consecutions in linear temporal logics and discuss the distinction between these two notions. The linear temporal logic $LDTL$, consisting of all formulas valid in the frame $(\mathbb{Z}, \leq, \geq)$ of all integer numbers, is the prime object of our investigation.

We describe consecutions admissible in $LDTL$ in a semantic way – via consecutions valid in special temporal Kripke/Hintikka models. Then we state that any temporal inference rule has a reduced normal form which is given in terms of uniform formulas of temporal degree 1. Using these facts and enhanced semantic techniques we construct an algorithm, which recognizes consecutions admissible in $LDTL$. Also, we note that using the same technique it follows that the linear temporal logic $L(N)$ of all natural numbers is also decidable w.r.t. inference rules. So, we prove that both logics $LDTL$ and $L(N)$ are decidable w.r.t. admissible consecutions. In particular, as a consequence, they both are decidable (known fact), and the given deciding algorithms are explicit.

1 Introduction

Tense logic was introduced by Arthur Prior (cf. [18, 19]) as a result of an interest in the relationship between tense and modality attributed to the Megarian philosopher Diodorus Cronus (ca. 340-280 B.C.). In subsequent years, temporal logic has received attention as an attractive approach towards the formalization of reasoning about time and knowledge. Initially the attention had been focused on the mathematical symbolic modelling of time (cf. [7]) and the finding of approaches to model temporal logic by varieties of temporal algebras and Kripke/Hintikka models (cf. S.K. Thomason [30]). Currently there are various temporal logics circumscribing multifarious properties of temporal information within a logical framework. They have numerous applications in AI and computer science (Clark et al., [5], Goldblatt [10]) and philosophy (van Benthem [31]).

Linear temporal logic is particular case of modal logic with linear alternative relations (cf. K. Segerberg [29]). In the general case, temporal logic can be viewed as a particular case of multi-modal logic (cf. Gabbay et al., [8]). Linear temporal logic LTL naturally models the steps in computational processes, therefore LTL has been quite successful in dealing
with applications to systems specification, verification (cf. Pnueli, [17]) and model checking (cf. [1, 5]). There are various approaches concerning the development of the mathematical theory of temporal logics and their applications to AI, CS and various enrichments of the temporal language (for instance, two-dimensional temporal logics (cf. [15])). From the mathematical point of view, the attention was focused primarily on two issues: decidability and the various forms of completeness and incompleteness. In this paper we will study the question about decidability of the linear temporal logics, but decidability in more general form – decidability w.r.t. admissible logical consecutions.

The study of logical inference constitutes a core of the construction of any logical system. Usually the study is based upon a choice of an axiomatic system (a set of axioms and inference rules). Then logical inference can be described in terms of derivations, (or if it is possible by some deduction theorems). The notion of inference depends upon the choice of the axiomatic systems, so, it is not an invariant. If a logic $\mathcal{L}$ is generated in a semantic way (a logic is just a set of all formulas valid in specified models) then there is no way to conclude which inferences are correct in $\mathcal{L}$. However, there is a way to define an invariant notion of logical inference, the so-called admissible rules (or, in other terms, consecutions). As far as the author is aware, the first reference to admissible rules is made by P. Lorenzen in [14]. Such rules are defined by the set of theorems of a given logic using the notion of (uniform) substitution and do not depend on the choice of axiomatic systems.

Admissible rules have been investigated for numerous modal and superintuitionistic logics. The history of the subject dates back to H. Friedman’s question [6] concerning existence of an algorithm which could distinguish rules admissible in intuitionistic propositional logic IPC, and also Harrop’s examples [11] of rules admissible in intuitionistic propositional logic IPC but not derivable in the Heyting axiomatic system for IPC. A strong result in this direction is the finding of sufficient conditions for derivable rules to be admissible in IPC (cf. G. Mints [16]). H. Friedman’s question was affirmatively solved by V. Rybakov [20], and later S. Ghilardi [9] found another solution.

In 1973 A. Kuznetsov raised the question whether IPC has a finite basis for admissible rules. The first positive results regarding this question were obtained by A. Citkin [3] who found a basis for all quasi-characteristic rules admissible in IPC. But A. Kuznetsov’s question was solved in the negative by Rybakov in [21], where it was proved that IPC does not have bases for admissible rules in finitely many variables. Later on Iemhoff [13] found an infinite explicit basis for rules admissible in IPC and constructed a characterization of IPC by means of rules from this basis [12]. Earlier an implicit basis – a recursive infinite basis – was proposed by Rybakov, Terziler, Rimazki [26]. An explicit basis for rules admissible in the modal logic $S4$ was found by Rybakov [27].

It was discovered that inference rules in a generalized form (inference rules with meta-variables) allow a description of logically unifiable formulas. Rybakov [22] found an algorithm which for any inference rule with meta-variables determines whether this rule is admissible in IPC. This algorithm can check logical unification in IPC and verify the solvability of logical equations. Later S. Ghilardi [9] discovered a new approach to unification in IPC via projective formulas. Similar results for inference rules with meta-variables admissible in modal transitive logics are presented in (Rybakov, [24]). A logic $\mathcal{L}$ is structurally complete if the class of all rules admissible in $\mathcal{L}$ and the class of all rules derivable in $\mathcal{L}$ co-
incide. Using ideas of W.Dziobiak, a complete characterization of hereditarily structurally complete modal logics extending $K4$ was found by Rybakov [23]. It was established that there is a big difference between refutation of formulas and refutation of inference rules. For instance in (Rybakov, Kiyatkin, Oner, [25]) it was shown that the majority of the logics with the finite model property (fmp) which extend IPC or the modal logic S4 do not have the fmp w.r.t. admissible inference rules. The technique developed for description of rules admissible in modal logics works for specific rules of common knowledge logics too (cf. [28]). For the case of temporal logics, relatively few results concerning admissible inference rules are known. In these logics it is more difficult to construct decision procedures.

In this paper we focus attention on the linear temporal logic $LDTL$ consisting of all formulas valid in the frame $\langle \mathbb{Z}, \leq, \geq \rangle$ of all integer numbers. In the beginning we describe rules admissible in $LDTL$ by a semantic way – via rules valid in special temporal Kripke/Hintikka models. Then we state that any temporal rule has reduced normal form which is given in terms of formulas of temporal degree 1. Using these facts and enhanced semantic techniques we find an algorithm which recognizes rules admissible in $LDTL$. Next, we note that using the same technique it is possible to construct deciding algorithms for the temporal logic $L(\mathbb{N})$ of all natural numbers. So $L(\mathbb{N})$ is also decidable w.r.t. inference rules. In the sequel we prefer the term consecution rather than inference rule in order to emphasize that we are interested in direct logical consequence, that is, to know what follows immediately from assumptions.

2 Notation, Preliminaries

We will use standard notation and known facts concerning modal and temporal logics and Kripke/Hintikka semantics (cf. [2, 24]). All necessary definitions are briefly presented below. A temporal frame is a pair $\mathcal{F} := \langle W, R \rangle$, where $W$ is a set (whose elements represent temporal states) and $R$ is a binary relation on $W$ (which imitates the flow of time). We will represent temporal logic as bi-modal logic, which is a standard approach. The language of the temporal logic consists of propositional letters, Boolean logical operations, and two modal operations: $\Box_+, \Box_-$. The formation rules for wff’s are standard, and $\Box_+ \Box_- A$ has the meaning $A$ will always be true, $\Box_- A$ has the meaning $A$ has always been true.

The temporal frames can be represented as triples $\mathcal{F} := \langle W, R, R^{-1} \rangle$, where $R^{-1}$ is the inverse of $R$. We can also consider as temporal frames the pairs $\mathcal{F} := \langle W, R \rangle$ – bearing in mind the presence of the inverse to relation $R$. Formulas $\Diamond_+ \Box_- A$ and $\Diamond_- \Box_+ A$ are abbreviations for $\neg \Box_- \neg A$ and $\neg \Box_+ \neg A$. Also we set $\Box A := A \land \Box_+ A \land \Box_- A$. For a frame $\mathcal{F}$, $L(\mathcal{F})$ denotes the temporal logic generated by $\mathcal{F}$ (i.e. $L(\mathcal{F})$ is the set of all formulas which are true in $\mathcal{F}$ w.r.t. all valuations).

The logic $LDTL$ (linear discrete temporal logic) is the set of all formulas which are true in the bi-frame $\mathcal{Z} := \langle \mathbb{Z}, \leq, \geq \rangle$ consisting of all integer numbers with their usual order-relations $\leq$ and $\geq$, i.e. $LDTL := L(\mathcal{Z})$. The frame $\mathcal{Z}$ is an intuitively well motivated model for the flow of time, it matches well with the human perception of time flow. An early axiomatization for this logic was proposed by K. Segerberg [29]. It consists of the
Hilbert style axiomatic system for the bi-modal logic $K$, together with the following axioms

\[ T : \Box_+ A \rightarrow A, \ \Box_- A \rightarrow A, \]
\[ Dum_+ : \Box_+ (\Box_+ (A \rightarrow \Box_+ A) \rightarrow A) \rightarrow (\Box_+ \Box_+ A \rightarrow A), \]
\[ Dum_- : \Box_- (\Box_- (A \rightarrow \Box_- A) \rightarrow A) \rightarrow (\Box_- \Box_- A \rightarrow A), \]
\[ C_0 : \Box_- \Box_+ A \rightarrow A, \]
\[ C_1 : \Box_+ \Box_- A \rightarrow A, \]
\[ 4 : \Box_+ A \rightarrow \Box_+ \Box_+ A, \ \Box_- A \rightarrow \Box_- \Box_- A, \]
\[ L_0 : \Box A \rightarrow \Box_+ \Box_+ A, \]
\[ L_1 : \Box A \rightarrow \Box_+ \Box_- A. \]

For a frame $F := \langle F, R \rangle$ with a reflexive and transitive $R$, a cluster $C$ is a subset of $F$ such that $\forall a, b \in C (aRb \land bRa)$ and $\forall a, b (a \in C \land b \in F \land aRb \land bRa \Rightarrow b \in C)$. In the sequel we will often denote clusters as $\bigcirc$ with some lower indexes. For a frame $F := \langle F, R \rangle$, we will use notation $a \in F$ if $a$ is a world (element) from the base set of $F$. A frame $F := \langle F, R \rangle$ is linear if, for any $a, b \in F$, $aRb$ or $bRa$. For any frame $F := \langle F, R \rangle$ and $a \in F$,

\[ a^{R+} := \{ b \mid b \in F, aRb \} \cup \{ b \mid b \in F, bRa \}. \]

A frame $F$ is quasi-linear if for any $a \in F$ $a^{R+}$ is linear. For any two transitive and reflexive frames $F_1 := \langle F_1, R_1 \rangle$ and $F_2 := \langle F_2, R_2 \rangle$, $F_1 \oplus F_2$ will denote the subsequent concatenation of $F_1$ and $F_2$; that is, the base set of $F_1 \oplus F_2$ is the disjoint union of $F_1$ and $F_2$, and the relation $R$ in $F_1 \oplus F_2$ is as follows: $R := R_1 \cup R_2 \cup \{(a, b) \mid a \in F_1, b \in F_2\}$.

For a model $M := \langle W, R, V \rangle$, $a \in W$ and any formula $\mathcal{B}$, $(M, a) \models \mathcal{B}$ is the notation for $\mathcal{B}$ is true at $a$ in the model $M$. If $M$ is clear from the context we just write $a \models \mathcal{B}$. The notation $M \models \mathcal{B}$ means $\mathcal{B}$ is valid in $M$, i.e. $(M, a) \models \mathcal{B}$ for all $a \in W$. If $F$ is a frame and $\mathcal{B}$ is a formula, $F \models \mathcal{B}$ is the notation for $(F, V) \models \mathcal{B}$ for all possible valuations $V$. If $F \models \mathcal{B}$, we say $\mathcal{B}$ is valid in $F$.

For any class of frames $\mathcal{K}$, the logic $\mathcal{L}(\mathcal{K})$, generated by $\mathcal{K}$, is the set of formulas $\{ A \mid \forall F \in \mathcal{K} (F \models \mathcal{A}) \}$. If $A \in \mathcal{L}(\mathcal{K})$, then we say $A$ is a theorem of $\mathcal{L}(\mathcal{K})$. For a frame $F$, $\mathcal{L}(F) := \mathcal{L}(\{ F \})$. For a logic $\mathcal{L}$, and a frame $F$, $F$ is said to be a $\mathcal{L}$-frame if $\mathcal{L} \subseteq \mathcal{L}(F)$.

Recall that, for a given logic $\mathcal{L}$, $\mathcal{L}$ is said to have the finite model property (fmp in the sequel) if, for any formula $A \notin \mathcal{L}$, there is a finite $\mathcal{L}$-frame $F$ such that $F \not\models A$. Usage of the fmp is a widely known approach to show that a propositional logic $\mathcal{L}$ is decidable. Usually, to show decidability, one employ the fmp with the effective upper bound.

Concerning $\text{LDTL}$, it is not easy to prove decidability because $\text{LDTL}$ does not have the finite model property. So, to prove the decidability, either a more powerful technique is employed (e.g. automata or the Rabin theorem, techniques are known) or a refinement of the conventional technique is needed. To start, we give our own proof that $\text{LDTL}$ does not have the fmp.
Theorem 2.1 The logic LDTL does not have the finite model property.

Proof. First, the formula

\[ A_0 := \neg[q \land \square_+(p \land \square_+q) \land \square_-(\neg p \land \square_+q) \land \square_+(p \land \square_+q) \land \square_-(\neg p \land \square_+q)] \]

is invalid in LDTL because \( Z \) has infinite ascending and descending chains. Indeed, if we take the valuation

\[ V(q) := Z \setminus \{0\}, \quad V(p) := \{2 \cdot i \mid i \in Z\}, \]

then it is easy to verify that \((Z, 0) \nvDash A_0\).

So, \( A_0 \notin LDTL \), but below we show that \( A_0 \) cannot be refuted by any finite LDTL-frame. It is easy to see that any LDTL-frame \( F := (W, R, R^{-1}) \) must be reflexive, transitive and quasi-linear. Therefore, if \( F \) is a finite LDTL-frame and it refutes \( A_0 \), then we can assume that \( F \) has the following structure: it has an \( R \)-smallest non-degenerate cluster, an \( R \)-greatest non-degenerate cluster and a non-empty linear part situated between the clusters. That is, \( F = O_1 \oplus Lin \oplus O_2 \), where \( O_i, i = 1, 2 \) are \( F \)-clusters, where each one has at least two worlds; and \( Lin \) is a linear non-empty frame. Next, the formula

\[ A_1 := \neg[q \land \diamond_+q \land \square_+(q \rightarrow \square_+q) \land \square_-(q \rightarrow \square_+q) \land \square_+(q \rightarrow [\square_+\diamond_+p \land \square_+\diamond_-(\neg p) \land [\square_+\diamond_-(p \land q) \land \square_+\diamond_-(\neg p \land q)]] \]

is satisfiable in \( F = O_1 \oplus Lin \oplus O_2 \). Indeed, fix an \( x \in O_2 \) and take the valuation:

\[ V(q) := O_2, \quad V(p) := \{x\}. \]

Simple conventional computation shows that \( \forall x \in Lin, (O_1 \oplus Lin \oplus O_2, x) \nvDash A_1 \).

Lemma 2.2 \( \neg A_1 \in LDTL \).

Proof. Assume that there is a valuation \( V \) for \( q \) and \( p \) such that \((Z, m) \nvDash V A_1 \) for some \( m \in Z \). Then, in particular, \((Z, m) \nvDash V \neg q \). Next, \((Z, m) \nvDash V A_1 \) yields that there exists \( k \), where \( m < k \) and \( k \nvDash V q \). Take the minimal \( k \) with this property. We have \( \forall n, n < k \Rightarrow n \nvDash V q \) for any \( n \geq m \) or \( n \leq m \), because \((Z, m) \nvDash V A_1 \). In particular, since \((Z, m) \nvDash V A_1 \),

\[ k \nvDash V \square_+\diamond_-(p \land q) \land \square_+\diamond_-(\neg p \land q), \]

which is impossible, because \( k \) is smallest among all \( y \in Z \) where \( y \nvDash V q \). \( \square \)

Thus, \( F \) is not an LDTL-frame, which concludes the proof of our theorem. \( \square \)

Consider the temporal frame: \( N := (N, \leq, \geq) \) consisting of all natural numbers with the usual order relations \( \leq \) and \( \geq \) on \( N \). The proof of the previous theorem can easily be modified to show that
Theorem 2.3  The logic $\mathcal{L}(N)$ does not have the finite model property.

So, linear temporal logics are quite different from linear modal logics, which usually possess the FMP with effective upper bounds.

3  Admissible Consecutions, Discussion, Preliminary Facts, Notation

In general terms, the problem of logical inference can be formulated as follows: given a set of assumptions $A_1, \ldots, A_n$, deduce what are the correct logical consequences $B$ from these assumptions. To fix notation, the writing of any formula in the displayed form $A(p_1, \ldots, p_n)$ signifies that this formula contains letters $p_1, \ldots, p_n$.

For a collection $A_1(x_1, \ldots, x_n), \ldots, A_m(x_1, \ldots, x_n), B(x_1, \ldots, x_n)$ of formulas, where $x_1, \ldots, x_n$ are all letters occurring in these formulas, the expression

$$
cs := A_1(x_1, \ldots, x_n), \ldots, A_m(x_1, \ldots, x_n) / B(x_1, \ldots, x_n),
$$

(or for short, $cs := A_1(x_1, \ldots, x_n), \ldots, A_m(x_1, \ldots, x_n)/B(x_1, \ldots, x_n)$) is called a consecution or an inference rule. Formulas $A_1(x_1, \ldots, x_n), \ldots, A_m(x_1, \ldots, x_n)$ are the premises (or assumptions) and $B(x_1, \ldots, x_n)$ is the conclusion of $cs$. The informal meaning of $cs$ is: $B(x_1, \ldots, x_n)$ is a logical consequence of formulas $A_1(x_1, \ldots, x_n), \ldots, A_n(x_1, \ldots, x_n)$. Note that any formula $A$ can be viewed as the consecution $\top/A$ with always true premise.

In the formulas above, the letters $x_i$ have a special meaning. We will analyze substitutional examples of these formulas, where $x_i$ can be replaced by arbitrary formulas. So, the letters $x_i$ are actually variables, therefore we use $x_i$ not $p_i$ in order to avoid confusion. For a formula $A(x_1, \ldots, x_n)$ and a collection of formulas $B_1, \ldots, B_n$, $A(B_1, \ldots, B_n)$ is the result of the replacement of all $x_i$ in $A(x_1, \ldots, x_n)$ by the formulas $B_i$.

The problem: what does logical consequence mean, is crucial. The question – which consecutions correctly model logical consequence for a given logic $\mathcal{L}$ – does not have an evident and definite answer. For a logic $\mathcal{L}$ with a fixed axiomatic system $Ax_\mathcal{L}$ and a consecution $cs := A_1, \ldots, A_n/B$, $cs$ is said to be derivable if $A_1, \ldots, A_n \vdash_{Ax_\mathcal{L}} B$. The derivable consecutions are safely correct.

For some logics $\mathcal{L}$ with axiomatic systems $Ax_\mathcal{L}$, it may happen that $A_1, \ldots, A_n \not\vdash_{Ax_\mathcal{L}} B$, but the rule $cs := A_1, \ldots, A_m/B$ is nevertheless correct for $\mathcal{L}$. That is, $cs$ derives $\mathcal{L}$-theorems from substitutional examples of the premises which are $\mathcal{L}$-theorems. Such rules were introduced by Lorenzen [14]: they are so-called admissible inference rules. The author found a reference to the issue [14] in ([4], p. 97).

For a propositional logic $\mathcal{L}$, a consecution

$$
cs := A_1(x_1, \ldots, x_n), \ldots, A_m(x_1, \ldots, x_n) / B(x_1, \ldots, x_n)
$$

is said to be admissible in $\mathcal{L}$ if, for any formulas $C_1, \ldots, C_n$,

$$
A_1(C_1, \ldots, C_n) \in \mathcal{L} \& \ldots \& A_m(C_1, \ldots, C_n) \in \mathcal{L} \text{ implies } B(C_1, \ldots, C_n) \in \mathcal{L}.
$$
So, \( \text{cs} \) is admissible in \( \mathcal{L} \) if \( \mathcal{L} \) (as the set of all its theorems) is closed w.r.t. \( \text{cs} \). An early example of a consecution which is admissible in the intuitionistic logic IPC but not derivable in the Heyting axiomatic system for IPC is Harrop’s rule (cf. [11]):

\[
q := \frac{\neg x \rightarrow y \lor z}{(\neg x \rightarrow y) \lor (\neg x \rightarrow z)}.
\]

That is, \( \neg x \rightarrow y \lor z \vdash_{\text{IPC}} (\neg x \rightarrow y) \lor (\neg x \rightarrow z) \), but, for any formulas \( A, B \) and \( C \), as soon as \( \vdash_{\text{IPC}} \neg A \rightarrow B \lor C \), it follows that \( \vdash_{\text{IPC}} (\neg A \rightarrow B) \lor (\neg A \rightarrow C) \).

The Mints consecution (cf. [16])

\[
\frac{(x \rightarrow y) \rightarrow x \lor y}{((x \rightarrow y) \rightarrow x) \lor ((x \rightarrow y) \rightarrow y)}
\]

is another example of a consecution which is non-derivable but admissible in IPC.

The Lemmon-Scott rule (cf. [24]),

\[
\frac{\Box ((\Box \Box p \rightarrow \Box p) \rightarrow (\Box p \lor \Box \neg p))}{\Box \Box p \lor \Box \neg p}
\]

is admissible but not derivable in modal logics S4, S4.1, Grz with conventional Hilbert-style axiomatizations.

In this paper we study logical consecutions of LDTL. As is commented above, we can accept as correct consecutions all derivable consecutions. And more, we can use all valid consecutions. Recall that, for a consecution \( \text{cs} := A_1, \ldots, A_n / B \), we say that \( \text{cs} \) is valid in a temporal logic \( \mathcal{L} \) if the formula \( \Box [\bigwedge_{1 \leq i \leq n} A_i] \land \bigwedge_{1 \leq i \leq n} A_i \rightarrow B \) occurs in \( \mathcal{L} \).

It is safe to accept these consecutions as correct, and, in particular, using the axiomatic system AX for LDTL proposed by Segerberg [29], it is easy to show that \( \text{cs} \) is valid in LDTL iff it is derivable (an appropriate deduction theorem works). Now we turn our attention to whether the consecutions admissible in LDTL are stronger than derivable (valid) ones. This is indeed the case - in particular, because LDTL possesses admissible but invalid (not derivable) consecutions (e.g. since it has non-unifiable but strongly satisfiable formulas).

We say that \( A(p_1, \ldots, p_n) \) is strongly satisfiable in a logic \( \mathcal{L} \) if there is a frame \( F \) and a valuation \( V \) in \( F \), where \( \forall b \in F \left[ ( F, b ) \models \neg A(p_1, \ldots, p_n) \right] \). And \( A(p_1, \ldots, p_n) \) is said to be unifiable in \( \mathcal{L} \) iff there are formulas \( B_1, \ldots, B_n \) which unify \( A(p_1, \ldots, p_n) \) in \( \mathcal{L} \): \( A(B_1, \ldots, B_n) \in \mathcal{L} \).

It is easy to see that, for any unifiable but strongly satisfiable formulas \( A_1, \ldots, A_m \), the consecution \( A_1, \ldots, A_m / \bot \) is invalid but admissible. For instance, the formula \( \Diamond \Box \neg p_i \land \Diamond \Box \neg p_i \) is non-unifiable but strongly satisfiable in LDTL. For any logic \( \mathcal{L} \), all consecutions admissible in \( \mathcal{L} \) (by their definition) form the greatest class of consecutions w.r.t. which \( \mathcal{L} \) is closed. Our aim is to prove that LDTL is decidable w.r.t. admissible consecutions. We need some techniques which are discussed in the next section.

### 4 Recognizing Consecutions Admissible in LDTL

We say a model \( \mathcal{M} := \langle M, R, V \rangle \) refutes a consecution \( \text{cs} := A_1, \ldots, A_n / B \) if

\[
\forall i, \forall w \in M((M, w) \models \neg V A_i) \land \exists g \in M((M, g) \models \Box V B).
\]
A consecution $cs$ is valid in $M$ if $M$ does not refute $cs$. A frame $F$ refutes $cs$ if a model based on $F$ refutes $cs$ (otherwise we say $cs$ is valid in $F$).

Writing a consecution in the form $cs(x_1, \ldots, x_k)$ signifies that all premises and the conclusion of $cs$ are built up on letters $x_1, \ldots, x_k$. For any consecution $cs$, $Sub(cs)$ is the set of all subformulas of the formulas from the premises and the conclusion of $cs$.

**Definition 4.1** For a temporal logic $L$ and a model $M$ with a valuation defined for a set of letters $p_1, \ldots, p_k$, $M$ is said to be $k$-characterizing for $L$ iff the following holds. For any formula $A(p_1, \ldots, p_k)$ in letters $p_1, \ldots, p_k$, $A(p_1, \ldots, p_k) \in L \iff M \models A(p_1, \ldots, p_k)$.

**Definition 4.2** For a given model $M$ with a valuation $V$, an another valuation $S$ of arbitrary letters $x_i$ in $M$ is said to be definable iff, for any $x_i$, there is a formula $A_i$ in letters from the domain of $V$ such that $S(x_i) = V(A_i)$.

In the sequel we will use the following simple fact (see, for instance, [24], p. 297).

**Lemma 4.3** A consecution $cs$ is not admissible in a logic $L$ iff, for any sequence of $k$-characterizing models, there are a number $n$ and $n$-characterizing model $Ch_L(n)$ from this sequence such that the frame of $Ch_L(n)$ refutes $cs$ by a certain valuation definable in $Ch_L(n)$.

We need a special sequence of $k$-characterizing models for $LDTL$. Usually, to construct $k$-characterizing models for modal and superintuitionistic logics, the finite model property has been used (cf. [20, 22, 24]). However, $LDTL$ does not have the fmp. Therefore we construct these models using infinite linear frames.

Let $M_k$ be the disjoint union of all models $⟨Z, V⟩$ based on $Z$, where $V$ are all possible valuations with $Dom(V) = \{p_1, \ldots, p_k\}$, and of all models based on the single reflexive element with all possible valuations of letters $p_1, \ldots, p_k$. The base sets of these models are evidently uncountable, $||M|| = 2^{ω}$.

**Theorem 4.4** The model $M_k$ is $k$-characterizing for $LDTL$.

**Proof.** This follows immediately from our semantic definition of $LDTL$.  □

For a given consecution $cs$, we say $cs$ has reduced normal form if

$$cs = \bigvee_{1 \leq j \leq m}(\bigwedge_{1 \leq i \leq n}[x_i^{k(j,i,0)} \land (\bigodot_+ x_i)^{k(j,i,1)} \land (\bigodot_- x_i)^{k(j,i,2)}]) \bigwedge x_1,$$

where $x_s$ are some letters, $k(i, j, z) \in \{0, 1\}$ and for any formula $C$, $C^0 := C, C^1 := \neg C$.

For a consecution in reduced normal form (in the sequel, denotation $cs_{nf}$ signifies that this consecution has reduced normal form), $cs_{nf}$ is said to be a normal reduced form of a consecution $cs$ iff, for any temporal logic $L$, $cs$ is admissible in $L$ iff $cs_{nf}$ is. Using exactly the same ideas as for Lemma 3.1.3 and Theorem 3.1.11 in [24] we can prove
Theorem 4.5 There exists an algorithm which, for any given consecution \(cs\), constructs its normal reduced form \(cs_{\text{nf}}\).

The analogue of this theorem for inference rules in modal logics was obtained in [20]. So, any modal inference rule has an equivalent reduced normal form of modal degree 1. As an immediate consequence of this fact, it was shown (Rybakov [24], Corollary 3.1.27) that any normal modal logic extending \(K4\) can be axiomatized by formulas of modal degree at most 2 (this is a known result proved first by Zakharyaschev). Using our technique it can be proved with ease as follows.

Take a modal formula \(A\). Its validity is equivalent to the validity of the consecution \(c := \top/\mathcal{A}\). Take its reduced normal form \(rf(c) = B/x_1\). For any consecution \(c_1 = G/D\), its transformation into the semi-universal formula \(f(c_1)\) is the formula \(\square G \land G \rightarrow D\). And, for any modal logic \(\mathcal{L}\) extending \(K4\), \(\mathcal{L} \oplus \mathcal{A} = \mathcal{L} \oplus f(rf(c))\).

In the case of temporal logics, using the transformation of consecutions into reduced normal forms described above, we can obtain a similar result: any temporal logic enriched with the universal modality (note: transitivity of time relations is not required) can be axiomatized by formulas of modal/temporal degree at most 2. The same holds for simply temporal logics where \(\square_+\) and \(\square_-\) are \(K4\)-modalities.

For any consecution \(cs_{\text{nf}}\) in normal reduced form, \(Pr(cs_{\text{nf}})\) is the premise of \(cs_{\text{nf}}\) and \(D(cs_{\text{nf}})\) is the set of all disjuncts from \(Pr(cs_{\text{nf}})\). \(Var(cs_{\text{nf}})\) is the set of all letters from \(cs_{\text{nf}}\). For any \(D \in D(cs_{\text{nf}})\), \(Sub(D)\) is the set of all subformulas of \(D\).

For any \(D \in D(cs_{\text{nf}})\), we set

\[
\begin{align*}
Pos^+(D) &= \{x_j \mid x_j \in Var(cs_{\text{nf}}), \neg \Diamond_+ x_j \notin Sub(D)\}; \\
Pos^-(D) &= \{x_j \mid x_j \in Var(cs_{\text{nf}}), \neg \Diamond_- x_j \notin Sub(D)\}; \\
D_{\text{now}} &= \{x_j \mid x_j \in Var(cs_{\text{nf}}), \neg x_j \notin Sub(D)\}; \\
\forall \mathcal{Y} \subseteq D(cs_{\text{nf}}), \text{ } Pos^+(\mathcal{Y}) := \bigcup_{D \in \mathcal{Y}} Pos^+(D); \text{ } Pos^-(\mathcal{Y}) := \bigcup_{D \in \mathcal{Y}} Pos^-(D).
\end{align*}
\]

Lemma 4.6 For any model \(M := \langle M, R, \mathcal{V} \rangle\), where \(M \models \mathcal{V} Pr(cs_{\text{nf}})\), for any \(a \in M\), there is a unique disjunct \(D\) from \(D(cs_{\text{nf}})\) such that \((M, a) \models \mathcal{V} D\).

In the sequel we will denote this unique disjunct by \(D_M(a)\).

In what follows, writing a frame (a model) in the form \(\Diamond_- \oplus \mathcal{F} \oplus \Diamond_+\) signifies that the frame (the frame of the model) is the sequential composition of a cluster \(\Diamond_-\), a frame \(\mathcal{F}\), which is a nonempty linear frame, and a cluster \(\Diamond_+\).

Lemma 4.7 If \(cs_{\text{nf}}\) is not admissible in \(LDTL\), then there is a finite model \(M\) of size at most \(6 \ast ||D(cs_{\text{nf}})|| + 2\) refuting \(cs_{\text{nf}}\) such that:

(i) \(M_{cs_{\text{nf}}} = \Diamond_- \oplus [a_1, \ldots, a_m] \oplus \Diamond_+\), where \([a_1, \ldots, a_m]\) is a collection of integer numbers with standard linear ordering \(\leq\), and \(\Diamond_-\), \(\Diamond_+\) are some clusters;
(ii) \( \forall x \in \bigcirc (Pos^+(D_M(x)) = Pos^+(D_M(a_1)); \)

(iii) \( \forall x \in \bigcirc (Pos^-(D_M(x)) = Pos^-D_M(a_m)) \), and

(iv) There is a valuation \( V \) for all letters \( x \) from \( cs_{nf} \) at the reflexive single element
frame \( \odot \), where \( \odot \models V Pr(cs_{nf}) \).

Proof. Since \( cs_{nf} \) is not admissible, some substitution \( sub : x_i \rightarrow C_i \) turns all its premises into theorems of \( LDTL \). In particular, the application of \( sub \) to the premises makes all of them valid in the frame \( \odot \), thus (iv) holds. Next, for some \( k \)-characterizing model \( M_k \) for \( LDTL \), \( cs_{nf} \) is refuted in \( M_k \) by some definable valuation (cf. Lemma 4.3). Consequently, \( cs_{nf} \) is refuted in some disjoint component \( Z \) of \( M_k \) by a valuation \( V \). Then there are elements \( a_1 \in Z \) where \( a_1 \models V \neg x_1 \). Fix some such \( a_1 \). Take any number \( a_{min} > a_1 + 1 \), where

\[
\forall x \in Z, Pos^+(D_Z(a_{min})) \subseteq Pos^+(D_Z(x)).
\]

(1)

Take the minimal number \( a^{max}_{min} \) such that

\[
\forall x \geq a_{min}, Pos^-(D_Z(a^{max}_{min})) \supseteq Pos^-(D_Z(x)).
\]

(2)

Next, we choose analogous numbers in the opposite direction. So, first take a number \( a_{min} < a_1 - 1 \), where

\[
\forall x \in Z, Pos^-(D_Z(a_{min})) \subseteq Pos^-(D_Z(x)).
\]

(3)

Then take the maximal number \( a^{min}_{max} \) such that

\[
\forall x \leq a_{min}, Pos^+(D_Z(a^{min}_{max})) \supseteq Pos^+(D_Z(x)).
\]

(4)

For any \( x, y \in Z \) with \( x < y \) we define the model \( Rep(x, y) \) as follows. For any \( D \in \{D_Z(a) \mid x < a < y \} \) choose the maximal element \( max_D \) from \([x, y]\), where \( (Z, max_D) \models V D \) and also the minimal element \( min_D \) from \([x, y]\), where \( (Z, min_D) \models \neg V D \). The base set of the model \( Rep(x, y) \) consists of \( x, y \) and all numbers \( max_D \) and \( min_D \) chosen above. The relation \( \leq \) on \( Rep(x, y) \) is the standard order on numbers. The valuation \( V \) is simply transferred into \( Rep(x, y) \) from \( Z \).

For any \( x, y \in Z \) where \( x < y \), the model \( Thin(Z, x, y) \) is the model based on the set
\((-\omega, x - 1] \cup Rep(x, y) \cup [y + 1, +\omega) \) with the standard relation \( \leq \) and the valuation \( V \) taken
from \( Z \).

Lemma 4.8 For any \( a \in Z \cap Thin(Z, x, y) \), and \( D_Z(a) \),

\[
(Thin(Z, x, y), a) \models V D_Z(a); \quad \|Thin(x, y)\| \leq 2 \ast \|D(cs_{nf})\|.
\]
Proof. The claim follows by conventional inductive computation. □

Take the model Thin(ℤ, \text{a}_{\text{min}}^{\text{max}}, \text{a}_{\text{min}}^{\text{max}}). This model refutes \text{cs}_{\text{nf}}:

\[ \text{Thin}(ℤ, \text{a}_{\text{min}}^{\text{max}}, \text{a}_{\text{min}}^{\text{max}}) \not\models V_{\text{cs}_{\text{nf}}}. \] (5)

This follows from Lemma 4.8, because by the construction of Thin(ℤ, \text{a}_{\text{min}}^{\text{max}}, \text{a}_{\text{min}}^{\text{max}}) this model contains the world max_{z(a_1)}.

Now we modify the model \( Q := \text{Thin}(ℤ, \text{a}_{\text{min}}^{\text{max}}, \text{a}_{\text{min}}^{\text{max}}) \) as follows. Take the model

\[ \mathcal{M} := \bigcirc_-(ℤ, (-\omega, \text{a}_{\text{min}}^{\text{max}} - 1)) \oplus \text{Rep}(ℤ, \text{a}_{\text{min}}^{\text{max}}, \text{a}_{\text{min}}^{\text{max}}) \oplus \bigcirc_+(ℤ, [\text{a}_{\text{min}}^{\text{max}} + 1, +\omega]), \]

where \( \text{Rep}(ℤ, \text{a}_{\text{min}}^{\text{max}}, \text{a}_{\text{min}}^{\text{max}}) \) is the model defined above and the other components have the following structure.

For any \( k \in (-\omega, \text{a}_{\text{min}}^{\text{max}} - 1] \) take the maximal element \( e_{\text{max}}(k) \in (-\omega, \text{a}_{\text{min}}^{\text{max}} - 1] \) with \( D_Q(k) = D_Q(e_{\text{max}}(k)) \). All elements \( e_{\text{max}}(k) \) form the set \( \bigcirc_-(ℤ, (-\omega, \text{a}_{\text{min}}^{\text{max}} - 1)) \), and we set the accessibility relation \( R \) on this set by making it a cluster: \( xRy \) for all pairs \( x, y \). The valuation \( V \) on \( \bigcirc_-(ℤ, (-\omega, \text{a}_{\text{min}}^{\text{max}} - 1)) \) is transferred from \( ℤ \).

The structure of \( \bigcirc_+(ℤ, [\text{a}_{\text{min}}^{\text{max}} + 1, +\omega]) \) is similar. That is, for any \( k \in [\text{a}_{\text{min}}^{\text{max}} + 1, +\omega) \), we take the minimal element \( e_{\text{min}}(k) \in [\text{a}_{\text{min}}^{\text{max}} + 1, +\omega) \) where \( D_Q(k) = D_Q(e_{\text{min}}(k)) \). Denote the set of all such elements \( e_{\text{min}}(k) \) by \( \bigcirc_+(ℤ, [\text{a}_{\text{min}}^{\text{max}} + 1, +\omega]) \), and take the accessibility relation \( R \) on this set by postulating \( xRy \) for all pairs \( x, y \). And again the valuation \( V \) on \( \bigcirc_-(ℤ, [\text{a}_{\text{min}}^{\text{max}} + 1, +\omega]) \) is transferred directly from \( ℤ \).

Using (1), (2), (3), (4) and conventional computation on the truth values of formulas it follows that

\[ (\mathcal{M}, e_{\text{min}}(k)) \models V D_Q(e_{\text{min}}(k)); \quad (\mathcal{M}, e_{\text{max}}(k)) \not\models V D_Q(e_{\text{max}}(k)), \] (6)

\[ \forall k \in [\mathcal{M} \setminus \bigcirc_-(ℤ, (-\omega, e_{\text{min}}^{\text{max}} - 1)) \cup \bigcirc_+(ℤ, [e_{\text{min}}^{\text{max}} + 1, +\omega])], \]

\[ (\mathcal{M}, k) \not\models V D_Q(k). \] (7)

Using (5), (6), and (7), we conclude

\[ \mathcal{M} \not\models V_{\text{cs}_{\text{nf}}}. \] (8)

So, we constructed a finite model of size at most \( 6 + |D(c_f_{\text{cs}})| + 2 \), linear in the size of \( \text{cs}_{\text{nf}} \), refuting \( \text{cs}_{\text{nf}} \) and possessing all required properties (i), (ii) and (iii). □

**Lemma 4.9** If, for a consecution \( \text{cs}_{\text{nf}} \), there is a finite model \( \mathcal{M}_{\text{cs}_{\text{nf}}} := \langle \mathcal{M}, V \rangle \) refuting \( \text{cs}_{\text{nf}} \) and satisfying properties (i), (ii), (iii) and (iv) from Lemma 4.7, then \( \text{cs}_{\text{nf}} \) is not admissible in LDTL.
Proof. By Lemma 4.3, it is sufficient to find a definable valuation \( S \) for the letters of \( cs_{\text{nf}} \) in some \( k \)-characterizing model \( \mathcal{M}_k \) for \( LDTL \), where \( S \) refutes \( cs_{\text{nf}} \). We choose \( k := |\text{Var}(cs_{\text{nf}})| \).

For any \( c \in \mathcal{M}_{cs_{\text{nf}}} \), \( D(c) \) is the disjunct member from \( D(cs_{\text{nf}}) \) which is true at \( c \) in the model \( (\mathcal{M}_{cs_{\text{nf}}}, V) \). By (i) from Lemma 4.7, \( \mathcal{M}_{cs_{\text{nf}}} = \bigcirc_+ \oplus [a_1, \ldots, a_m] \oplus \bigcirc_+ \).

We define the valuation \( S \) in \( \mathcal{M}_k \) as follows:

\[
\forall x_j \in \text{Var}(cs_{\text{nf}}), \quad S(x_j) := V(\bigvee \{D(c) \mid c \in \mathcal{M}_{cs_{\text{nf}}}, \neg x_j \notin \text{Sub}(D(c))\}). \tag{9}
\]

**Lemma 4.10** The definable valuation \( S \) refutes \( cs_{\text{nf}} \) in a disjoint component \( Z_1 \) of \( \mathcal{M}_k \).

Proof. Let \(|\bigcirc_-| = \{c_1^-, \ldots, c_n^-\} \) and \(|\bigcirc_+| = \{c_1^+, \ldots, c_n^+\} \). For any element \( a \) of any model with a valuation \( \mathcal{G} \), \( \text{Val}_\mathcal{G}(a) := \{p_i \mid p_i \in \text{Dom}(\mathcal{G}), a \models g p_i\} \). Consider the disjoint component \( Z_1 \) from \( \mathcal{M}_k \) with the following structure:

\[
Z_1 := \{w_i \mid i \in (-\omega, -1] \} \oplus [b_1, \ldots, b_m] \oplus \{v_i \mid i \in [1, +\omega)\}
\]

where \( w_i \leq w_j \) if \( i \leq j \), \( b_i \leq b_j \) if \( i \leq j \), and \( v_i \leq v_j \) if \( i \leq j \), and the valuation \( V \) works as follows:

\[
\text{Val}_V(b_j) = \text{Val}_V(a_j); \tag{10}
\]

\[
\forall i \in N, \forall j(1 \leq j \leq n^- - 1) \text{Val}_V(w_{-(i+n^-+j)}) = \text{Val}(c_j^-); \tag{11}
\]

\[
\forall i \in N, \forall j(1 \leq j \leq n^+ - 1) \text{Val}_V(v_{i+n^++j}) = \text{Val}(c_j^+). \tag{12}
\]

The existence of such \( Z_1 \) follows from the construction of \( \mathcal{M}_k \). Using (11), (12), (10) and (ii), (iii) from Lemma 4.7 we immediately derive,

\[
\forall v_{i+n^++j} \in Z_1, \forall w_{-(i+n^-+j)} \in Z_1, \forall x_j \in \text{Var}(cs_{\text{nf}}),
\]

\[
(Z_1, v_{i+n^++j}) \models \neg \vee \bigodot + x_j \Leftrightarrow (\mathcal{M}_{cs_{\text{nf}}}, c_j^+) \models \neg \vee \bigodot + x_j; \tag{13}
\]

\[
(Z_1, v_{i+n^++j}) \models \neg \vee \bigodot - x_j \Leftrightarrow (\mathcal{M}_{cs_{\text{nf}}}, c_j^-) \models \neg \vee \bigodot - x_j; \tag{14}
\]

\[
(Z_1, w_{-(i+n^-+j)}) \models \neg \vee \bigodot + x_j \Leftrightarrow (\mathcal{M}_{cs_{\text{nf}}}, c_j^-) \models \neg \vee \bigodot + x_j; \tag{15}
\]

\[
(Z_1, w_{-(i+n^-+j)}) \models \neg \vee \bigodot - x_j \Leftrightarrow (\mathcal{M}_{cs_{\text{nf}}}, c_j^-) \models \neg \vee \bigodot - x_j; \tag{16}
\]

\[
(Z_1, b_j) \models \neg \vee \bigodot - x_j \Leftrightarrow (\mathcal{M}_{cs_{\text{nf}}}, a_j) \models \neg \vee \bigodot - x_j. \tag{17}
\]

From (15), (16), (13), (14), (17), (10), (11) and (12) it immediately follows that

\[
(Z_1, w_{-(i+n^-+j)}) \models \neg \vee D(c_j^-); \tag{18}
\]

\[
(Z_1, v_{i+n^++j}) \models \neg \vee D(c_j^+); \tag{19}
\]

12
If Lemma 4.11 model the disjoint component \( Z \) then, for any \( e \) or \( g \) and \( h \) it follows that \( g \in e \) and \( e \in \mathcal{M} \) together with previous results and \( \mathcal{M} \) of the model \( M \).

Thus, \( (18), (19), (20) \) and \( (21) \) imply that the definable valuation \( S \) disproofs \( \text{cs}_{\text{nf}} \) in the disjoint component \( Z \) of the model \( M \). \( \square \)

To complete Lemma 4.9, we need to show that the premise of \( \text{cs}_{\text{nf}} \) is valid in the whole model \( M \) w.r.t. a definable valuation \( S_1 \) coinciding with \( S \) on the model \( Z \) chosen above.

**Lemma 4.11** If \( Z_2 \) is a component of \( M_k \) and \( Z_2 \models \neg \forall \Gamma \), where

\[
\Gamma := \Box \bigvee \{ D(a) \mid a \in \mathcal{M}_{\text{cs}_{\text{nf}}} \} \land \Box \neg \bigvee \{ D(a) \mid a \in \mathcal{M}_{\text{cs}_{\text{nf}}} \},
\]

then, for any \( e \in Z_2 \), if \( e \models \neg V D(a) \) then \( e \models S D(a) \).

**Proof.** Let \( e \in Z_2 \) and \( e \models \neg V D(a) \). For any letter \( x_j \) from \( D(a) \), \( h \models V x_j \Rightarrow h \models S x_j \) for any \( h \) \( Z_2 \) by the definition of \( S \). Assume \( e \models V x_j \), then for some \( g \), \( e \leq g \), where \( g \models V x_j \). But, since \( e \models \neg \forall \Gamma \), we have \( g \models V D(b) \) for some \( D(b) \) and, as we noticed above, \( g \models V x_j \) implies \( g \models S x_j \), therefore \( e \models S V x_j \).

Conversely, assume \( e \models S V x_j \). Then \( e \leq g \) where \( g \models S x_j \). Again, since \( e \models \neg \forall \Gamma \), it follows that \( g \models V D(b) \) for some \( D(b) \); and \( g \models S x_j \) implies \( g \models V x_j \) and \( e \models V x_j \). Thus, \( e \models V x_j \Leftrightarrow e \models S x_j \). Similarly we can show that \( e \models V x_j \Rightarrow \neg S x_j \), which, together with previous results and \( e \models V D(a) \) yields \( e \models S D(a) \). \( \square \)

Next, we choose the following definable valuation \( S_1 \) for letters \( x_j \) from \( \text{cs}_{\text{nf}} \). Let \( \epsilon(x_j) := \top \) if \( \models \neg V x_j \) and \( \epsilon(x_j) := \bot \) otherwise, where \( \top \) and \( V_1 \) are taken from (iv) in Lemma 4.7, and

\[
S_1(x_j) := [V(\Gamma) \land S(x_j)] \cup [V(\neg \Gamma) \land V(\epsilon(x_j))].
\]

By Lemma 4.10, (18), (19) and (20) \( S_1 \) refutes \( \text{cs}_{\text{nf}} \) in \( Z_1 \) - a disjoint component of \( M_k \). By Lemma 4.11 and (iv) from Lemma 4.7 the premise of \( \text{cs}_{\text{nf}} \) is valid w.r.t. \( S_1 \) in any disjoint component of \( M_k \). So, the definable valuation \( S_1 \) refutes \( \text{cs}_{\text{nf}} \) in \( M_k \) and hence \( \text{cs}_{\text{nf}} \) is not admissible in \( \text{LDTL} \). \( \square \)

Immediately from Lemmas 4.7 and 4.9 we derive our main result.

**Theorem 4.12** The logic \( \text{LDTL} \) is decidable w.r.t. admissible consecutions.

Using exactly the same approach, technique and structure of proofs (actually the new proofs are simplified versions of the ones given above for Theorem 4.12) we can prove

**Theorem 4.13** The temporal logic of natural numbers \( \mathcal{L}(\mathcal{N}) \) is decidable w.r.t. admissible consecutions.
In particular, as a consequence, we immediately obtain that the logics $LDTL$ and $\mathcal{L}(\mathcal{N})$ are decidable (w.r.t. theorems), though neither possesses the finite model property.

As an algebraic consequence, it follows that quasi-equational theories of the free temporal algebras $F_\omega(LDTL)$ and $F_\omega(\mathcal{L}(\mathcal{N}))$ are decidable. It seems that the decidability of first-order theories of these free algebras, is an open question. An interesting open problem is the extension of the results in this paper to the temporal logic $LTL$ with the temporal operation $\textit{until}$.

The author thanks the anonymous referee for valuable comments and corrections which helped to improve this paper.

References


